JEE Advanced 2016 Paper-2

Mathematics

37. Ans: (B)

$$P=\begin{bmatrix}1&0&0\\4&1&0\\16&4&1\end{bmatrix}\Rightarrow P^2=\begin{bmatrix}1&0&0\\8&1&0\\16+32&8&1\end{bmatrix}$$

so, $P^3=\begin{bmatrix}1&0&0\\12&1&0\\16+32+48&12&1\end{bmatrix}$ (from the symmetry)

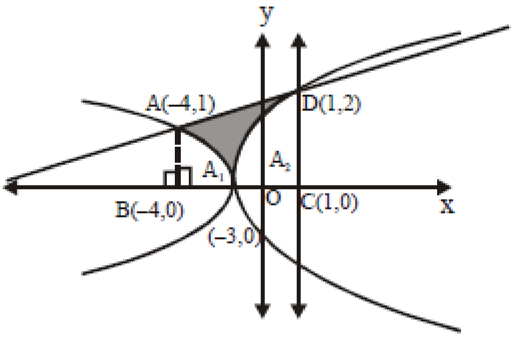
$$P^{50}=\begin{bmatrix}1&0&0\\200&1&0\\\frac{16.50.51}{2}&200&1\end{bmatrix}$$

As, $P^{50}-Q=I\Rightarrow q_{31}=\frac{16.50.51}{2}$

$q_{32}=200\,and\;q_{21}=200$

$\therefore\frac{q_{31}+q_{32}}{q_{21}}=\frac{16.50.51}{2.200}+1=102+1=103$

38. Ans: (C)



Clearly required area $=$ area (trapezium $ABCD$) $-\left(A_1+A_2\right)$ ...(i)

(trapezium $ABCD$) $=\frac{1}{2}(1+2)(5)=\frac{15}{2}$

$A_1=\int_{-4}^{-3}\sqrt{-(x+3)}dx$

$=\frac{2}{3}$

and $A_2=\int_{-3}^{1}(x+3)^{1/2}dx=\frac{16}{3}$

$\therefore$ From equation (1), we get required area $=\frac{15}{2}-\left(\frac{2}{3}+\frac{16}{3}\right)=\frac{3}{2}$

39. Ans: (C)

We have,

$$=2.\sum_{k=1}^{13}\frac{\sin\left(\left(\frac{k\pi}{6}+\frac{\pi}{4}\right)-\left((k-1)\frac{\pi}{6}+\frac{\pi}{4}\right)\right)}{\sin\left(\frac{\pi}{4}+(k-1)\frac{\pi}{6}\right).\sin\frac{\pi}{4}+\frac{\pi}{6}}=2\sum_{k=1}^{13}\left(\cot\left((k-1)\frac{\pi}{6}+\frac{\pi}{4}\right)-\cot\left(\frac{k\pi}{6}+\frac{\pi}{4}\right)\right)$$

$$=2\left[\cot\frac{\pi}{4}-\cot\left(\frac{13\pi}{6}+\frac{\pi}{4}\right)\right]=2\left(1-\cot\left(\frac{5\pi}{12}\right)\right)=2\left(1-\left(2-\sqrt{3}\right)\right)=2\left(\sqrt{3}-1\right)$$

40. Ans: (B)

If $\log_e b_1, \log_e b_2 \ldots \log_e b_{101} \rightarrow AP; D=\log_e 2$

$\Rightarrow b_1\, b_2\, b_3 \ldots b_{101} \rightarrow GP; r=2$

$\therefore b_1, 2b_1, 2^2 b_1, \ldots\ldots, 2^{100} b_1 \ldots GP$

$a_1\, a_2\, a_3 \ldots\ldots a_{101} \ldots AP$

Given, $a_1=b_1$ & $a_{51}=b_{51}$

$\Rightarrow \quad a_1+50D=2^{50}b_1$

$\therefore \quad a_1+50D=2^{50}a_1\left(\text{AS}\, b_1=a_1\right)$

Now, $t = b_1\left(2^{51}-1\right); s = \frac{51}{2}\left(2a_1 + 50D\right)$

$t = a_1.2^{51} - a_1 \Rightarrow t < a_1 \ldots (\text{i}) \quad ; \quad s = \frac{51}{2}\left(a_1 + a_1 + 50D\right)$

$s = \frac{51}{2}\left(a_1 + 2^{50}a_1\right)$

$s = \frac{51a_1}{2} + \frac{51}{2}.2^{50}a_1$

$\Rightarrow s > a_1.2^{51} \quad \ldots (\text{ii})$

clearly $s > t$ (from equation (i) and (ii))

Also $a_{101} = a_1 + 100D \quad ; \quad b_{101} = b_1.2^{100}$

$\therefore a_{101} = a_1 + 100\left(\frac{2^{50}a_1 - a_1}{50}\right) \quad ; b_{101} = 2^{100}a_1 \quad \ldots (\text{iii})$

$a_{101} = a_1 + 2^{51}a_1 - 2a_1 \Rightarrow a_{101} = 2^{51}a_1 - a_1 \Rightarrow a_{101} < 2^{51}a_1 \quad \ldots (\text{iv})$

clearly $b_{101} > a_{101} > a_{101}$ (from equation (iii) and (iv))

41. Ans: (A)

Let $I = \int_{-\pi/2}^{\pi/2} \frac{x^2 \cos x}{1+e^x} dx = \int_0^{\pi/2} \left(\frac{1}{1+e^x} + \frac{1}{1+e^{-x}}\right) x^2 \cos x \, dx$

$= \int_0^{x/2} x^2 \cos x \, dx = \left(x^2 \sin x\right)_0^{\pi/2} \quad -2\int_0^{\pi/2} x.\sin x dx$

$(I)(II) \qquad\qquad (I)(II)$

$= \frac{\pi^2}{4} - 2\left[-\left(x.\cos x\right)_0^{\pi/2} + \int_0^{\pi/2} 1.\cos x\right] = \frac{\pi^2}{4} - 2\left[0+1\right] = \left(\frac{\pi^2}{4} - 2\right)$

42. Ans: (C)

Line $AP : \frac{x-3}{1} = \frac{y-1}{-1} = \frac{z-7}{1} = \lambda$

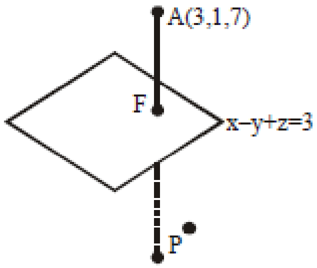
$\Rightarrow \quad F(3+\lambda, 1-\lambda, \lambda+7)$ lies in the plane

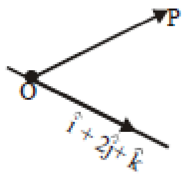
$\therefore \quad 3+\lambda-(1-\lambda)+\lambda+7=3$

$3\lambda=-6 \Rightarrow \lambda=-2$

$\Rightarrow \quad F(1,3,5)$

$\Rightarrow \quad P(-1,5,3)$





so required plane is $\begin{vmatrix} x-0 & y-0 & z-0 \\ 1 & 2 & 1 \\ -1 & 5 & 3 \end{vmatrix} = 0$

$\therefore x-4y+7z=0$

43. Ans: (A, B)

If $\quad x^3-x \geq 0 \quad \Rightarrow \quad \cos\left|x^3-x\right| = \cos\left(x^3-x\right)$

$x^3-x<0 \quad \Rightarrow \quad \cos\left|x^3-x\right| = \cos\left(x^3-x\right)$

Similarly, $b|x|\sin\left|x^3+x\right| = bx\sin\left(x^3+x\right)$ for all $x \in R$

$\therefore \quad f(x) = a\cos\left(x^3 - x\right) + bx\sin\left(x^3 + x\right)$

which is composition and sum of differentiable functions therefore always continuous and differentiable.

44. Ans: (B, C)

$$\ell n\, f(x) = \lim_{n\leftrightarrow\infty} \frac{x}{n}\,\ell n\left[\frac{\prod_{r=1}^{n}\left(x+\frac{1}{r/n}\right)}{\prod_{r=1}^{n}\left(x^2+\frac{1}{(r/n)^2}\right)}\frac{1}{\prod_{r=1}^{n}(r/n)}\right]$$

$$= x\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\ell n\left(\frac{x\frac{r}{n}+1}{\left(x\frac{r}{n}\right)^2+1}\right)$$

$$= x\int_0^1 \ell n\left(\frac{1+tx}{1+t^2x^2}\right)dt \qquad \text{put } tx = z \ \text{ sign scheme}$$

$$\ell n\, f(x) = \int_0^x \ell n\left(\frac{1+z}{1+z^2}\right)dz$$

$$\Rightarrow \frac{f'(x)}{f(x)} = \ell n\left(\frac{1+x}{1+x^2}\right)$$

sign scheme of $f'(x)$ $\quad \overset{+}{\longrightarrow}\ 1\ \overset{-}{\longrightarrow} \quad$ also $f'(1) = 0$

$$\Rightarrow f\left(\frac{1}{2}\right) < f(1),\ f\left(\frac{1}{3}\right) < f\left(\frac{2}{3}\right),\ f'(2) < 0$$

Also $\dfrac{f'(3)}{f(3)} - \dfrac{f'(2)}{f(2)} = \ell n\left(\dfrac{4}{10}\right) - \ell n\left(\dfrac{3}{5}\right)$

$$= \ell n\left(\frac{4}{6}\right) < 0 \Rightarrow \frac{f'(3)}{f(3)} < \frac{f'(2)}{f(2)}$$

45. Ans: (A, D)

Using $L'Hôpital's$ Rule

$$\lim_{x\to 2}\frac{f'(x)g(x)+f(x)g'(x)}{f''(x)g'+f'(x)g''(x)}=1$$

$$\Rightarrow \frac{f(2)g'(2)}{f''(2)g'(2)}=1\Rightarrow f'(2)f(2)>0$$

option (D) is right and option (C) is wrong

also $f'(2)=0$ and $f''(2)>0 \quad \therefore x=2$ is local minima.

46. Ans: (B, C)

$$|\hat{w}||\hat{u}\times\hat{v}|\cos\phi=1\Rightarrow\phi=0$$

$$\Rightarrow \hat{u}\times\hat{v}=\hat{w}\text{ also }|\vec{v}|\sin\theta=1$$

$\Rightarrow$ there may be infinite vectors $\vec{v}=\overrightarrow{OP}$

such that P is always 1 unit dist. from $\hat{u}$

For option (C) $\hat{u}\times\vec{v}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ u_1 & u_2 & 0\\ v_1 & v_2 & v_3\end{vmatrix}$

$$\hat{w}=(u_2v_3)\hat{i}-(u_1v_3)\hat{j}+(u_1v_2-u_2v_1)\hat{k}$$

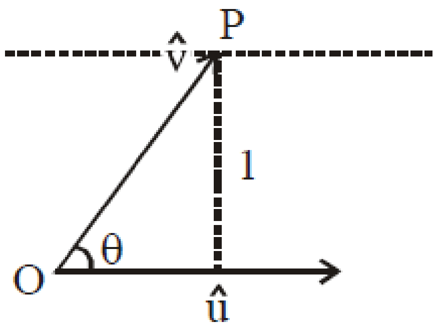
$$u_2v_3=\frac{1}{\sqrt{6}},-u_1v_3=\frac{1}{\sqrt{6}}\Rightarrow|u_1|=|u_2|$$

for option (D) : $\hat{u}\times\vec{v}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ u_1 & 0 & u_3\\ v_1 & v_2 & v_3\end{vmatrix}$

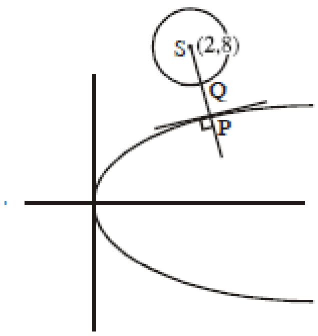
$$\hat{w}=(-v_2u_3)\hat{i}-(u_1v_3-u_3v_1)\hat{j}+(u_1v_2)\hat{k}$$

$$-v_2u_3=\frac{1}{\sqrt{6}},u_1v_2=\frac{2}{\sqrt{6}}$$

$\Rightarrow 2|u_3| = |u_1|$ So (D) is wrong



47. Ans: (A, C, D)



$y^2 = 4x$

point P lies on normal to parabola passing through centre of circle

$$y + tx = 2t + t^3 \quad \ldots(\text{i})$$
$$8 + 2t = 2t + t^3$$
$$t = 2$$
$$P(4,4)$$
$$SP = \sqrt{(4-2)^2 + (4-8)^2}$$
$$SP = 2\sqrt{5}$$
$$SQ = 2$$
$$\Rightarrow PQ = 2\sqrt{5} - 2$$
$$\frac{SQ}{QP} = -\frac{1}{\sqrt{5}-1} = \frac{\sqrt{5}+1}{4}$$

To find x intercept

put $y=0 \text{ in}(\text{i})$

$\Rightarrow \quad x=2+t^2$

$x=6$

$\therefore$ Slope of common normal $=-t=-2$

$\therefore$ Slope of tangent $=\frac{1}{2}$

48. Ans: (A, C, D)

$$x+iy=\frac{1}{a+ibt}$$

$$x+iy=\frac{a-ibt}{a^2+b^2t^2}$$

Let $a\neq 0 \,\&\, b\neq 0$

$$x=\frac{a}{a^2+b^2t^2} \quad \ldots(1)$$

$$y=\frac{-bt}{a^2+b^2t^2} \quad \ldots(2)$$

$$\frac{y}{x}=\frac{-bt}{a}\Rightarrow t=-\frac{ay}{bx}$$

Put in (1)

$$x\left\{a^2+b^2.\frac{a^2y^2}{b^2x^2}\right\}=a$$

$$a^2\left(x^2+y^2\right)=ax$$

$$x^2+y^2-\frac{1}{a}x=0$$

$$\left(x-\frac{1}{2a}\right)^2+y^2=\frac{1}{4a^2}$$

$\Rightarrow$ option (A) is correct

for $a\neq 0, b=0$

$$x+iy=\frac{1}{a}$$

$x=\frac{1}{a}, y=0 \Rightarrow z$ lies on x-axis $\Rightarrow$ option (C) is correct

for $a=0, b\neq 0$

$x+iy=\frac{1}{ibt}$

$y=-\frac{1}{bt}i, x=0$

$\Rightarrow z$ lies on y-axis. $\Rightarrow$ option (D) is correct

49. Ans: (B, C, D)

$ax+2y=\lambda$
$3x-2y=\mu$

for $a=-3$ above lies will be parallel or coincident

parallel for $\lambda+\mu\neq 0$ and coincident if $\lambda+\mu=0$

and if $a\neq -3$ lies are intersecting $\Rightarrow$ unique solution.

50. Ans: (B, C)

$f(x)=[x^2]-3$
$g(x)=(|x|+|4x-7|)([x^2]-3)$

$\therefore \quad f$ is discontinuous at $x=1,\sqrt{2},\sqrt{3},2$ in $\left[-\frac{1}{2},2\right]$

and $|x|+|4x-7|\neq 0$ at $x=1,\sqrt{2},\sqrt{3},2$

$\Rightarrow \quad g(x)$ is discontinuous at $x=1,\sqrt{2},\sqrt{3}$ in $\left(-\frac{1}{2},2\right)$

$(0-\delta, 0+\delta)$
In $g(x)=(|x|+|4x-7|).(-3)$
$\Rightarrow$ 'g' is non derivable at $x=0$.

In $\left(\frac{7}{4}-\delta, \frac{7}{4}+\delta\right)$

$g(x)=0$ as $f(x)=0$

$\Rightarrow$ Derivable at $x=\frac{7}{4}$

$\therefore$ 'g' is non-derivable at $0, 1, \sqrt{2}\,\frac{7}{4}$

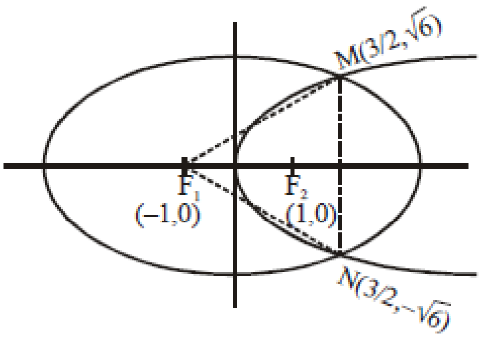
51. Ans: (B)

$$P(X>Y)=P(T_1\text{win})P(T_1\text{win})+P(T_1 win)P(\text{match draw})+P(T_1\text{win})$$
$$=\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{6}+\frac{1}{6}\times\frac{1}{2}=\frac{5}{12}$$

52. Ans: (C)

$$P(X=Y)=P(\text{match draw})P(\text{match draw})+P(T_1\text{win})P(T_2\text{win})+P(T_2\text{win})P(T_1\text{win})$$
$$=\frac{1}{6}\times\frac{1}{6}+\frac{1}{2}\times\frac{1}{3}+\frac{1}{3}\times\frac{1}{2}=\frac{13}{36}$$

53. Ans: (A)



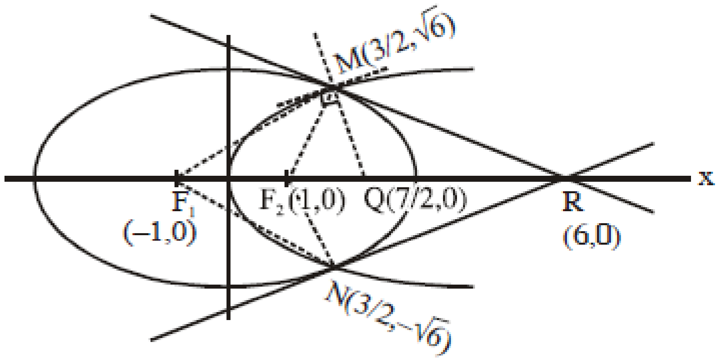
Orthocentre lies on x-axis

Equation of altitude through $M: y-\sqrt{6}=\frac{5}{2\sqrt{6}}\left(x-\frac{3}{2}\right)$

Equation of altitude through $F_1: y=0$

solving, we get orthocenter $\left(-\frac{9}{10},0\right)$

54. Ans: (C)



Normal to parabola at $M: r-\sqrt{6}=-\frac{\sqrt{6}}{2.1}\left(x-\frac{3}{2}\right)$

Solving it with $y=0$, we get $Q\equiv\left(\frac{7}{2},0\right)$

Tangent to ellipse at $M: \frac{x.\frac{3}{2}}{9}+\frac{y\left(\sqrt{6}\right)}{8}=1$

Solving it with $y=0$, we get $R\equiv(6,0)$

$\therefore$ Area of triangle $MQR=\frac{1}{2}.\left(6-\frac{7}{2}\right).\sqrt{6}=\frac{5\sqrt{6}}{4}$

Area of quadrilateral $MF_1NF_2=2.\frac{1}{2}.(1-(-1)).\sqrt{6}=2\sqrt{6}$

Required ratio $=5:8$